

Stability of Nonsingular Cosmologies in Galileons with Torsion

S. Mironov (INR RAS) & M. Valencia-Villegas (ITMP MSU)

Based on:

- S. Mironov and M. Valencia-Villegas, Phys. Rev. D, vol. 108, no. 2, p. 024057, 2023
- S. Mironov and M. Valencia-Villegas, 2307.06929

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Outline

1. Introduction: Galileons on a spacetime without Torsion
2. Galileons on a spacetime with Torsion
3. Torsionful Galileons on a Cosmological setting
4. Stability on a spacetime with Torsion
 - a. Example, a proof of principle: An almost always subluminal, all-time stable, non singular cosmology on a spacetime with torsion

1. Introduction:

Galileons on a spacetime

without Torsion

- **Definition**
- **Motivation**
- **Key Aspects**

1. Introduction: Galileons on a spacetime without Torsion

On top of GR, $\int d^4x \sqrt{-g} R$

consider four general functions $G_2(\phi, X)$, $G_3(\phi, X)$, $G_4(\phi, X)$, $G_5(\phi, X)$

with $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, notation: $G_{4,X} = \partial G_4/\partial X$, $(-, +, +, +)$.

Horndeski/ Generalized Galileons:

$$\begin{aligned} \mathcal{S} = \int d^4x \sqrt{-g} & \left(G_2 - G_3 \nabla_\mu \nabla^\mu \phi + G_4(\phi, X) R + G_{4,X} \left((\nabla_\mu \nabla^\mu \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right) \right. \\ & + G_5 G^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{G_{5X}}{6} \left((\nabla_\mu \nabla^\mu \phi)^3 - 3(\nabla_\mu \nabla^\mu \phi) (\nabla_\nu \nabla_\rho \phi) \nabla^\nu \nabla^\rho \phi \right. \\ & \left. \left. + 2(\nabla_\mu \nabla_\nu \phi) (\nabla^\nu \nabla^\rho \phi) \nabla^\mu \nabla_\rho \phi \right) \right) \end{aligned}$$

1. Introduction: Galileons on a spacetime without Torsion

Horndeski/ Generalized Galileons:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(G_2 - G_3 \nabla_\mu \nabla^\mu \phi + G_4(\phi, X) R + G_{4,X} \left((\nabla_\mu \nabla^\mu \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right) + G_5 \text{ part of the action} \right)$$

- No Ostrogradsky-Ghost
- *(No) NEC (See e.g. Rubakov, 2014) - (No P-H. Non-singular cosmologies)*
- Generality
- NO-GO (Libanov, Mironov and Rubakov, 2016), [10-17].
- Inspired by the low energy effective theory of DGP model [2].

2. Galileons on a spacetime

with Torsion

2. Galileons on a spacetime with Torsion.

Motivation to introduce Torsion in Galileons:

- 1. NO-GO: can a more general spacetime with torsion cure the linear stability issue?**

Answer so far: **yes, but other issues arise.**

Other general motivations:

2. Torsion is suggested by demanding local Poincaré invariance [18].
3. Torsion has also been studied in relation to nonsingular cosmologies (before Horndeski) [18].
4. Torsion is on the way to introduce spinors [18].

2. Galileons on a spacetime with Torsion.

- **Quartic Galileons with Torsion**

$$\mathcal{S}_4 = \int d^4x \sqrt{-g} \left(G_4(\phi, X) R + G_{4,X} \left((\nabla_\mu \nabla^\mu \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right) \right)$$

$$\nabla \rightarrow \tilde{\nabla}, \quad \tilde{\Gamma}_{\mu\lambda}^\nu \neq \tilde{\Gamma}_{\lambda\mu}^\nu \quad \left[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu \right] \phi \neq 0$$

$$\mathcal{S}_{4c} = \int d^4x \sqrt{-g} \left(G_4(\phi, X) \tilde{R} + G_{4,X} \left(\left(\tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi \right)^2 - \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right) \tilde{\nabla}^\nu \tilde{\nabla}^\mu \phi - c \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right) \left[\tilde{\nabla}^\mu, \tilde{\nabla}^\nu \right] \phi \right) \right),$$

c parameterises a family of theories with different dynamics.

Follows from: $G_{4,X} \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right)^2$?

$$G_{4,X} \left(\left(\tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi \right)^2 + c \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right) \tilde{\nabla}^\mu \tilde{\nabla}^\nu \phi + s \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right) \tilde{\nabla}^\nu \tilde{\nabla}^\mu \phi \right), \quad c + s = -1$$

2. Galileons on a spacetime with Torsion.

- **Explicit Torsion**

We introduce torsion in the metric (second order) formalism:

$$T^\rho{}_{\mu\nu} = \tilde{\Gamma}^\rho{}_{\mu\nu} - \tilde{\Gamma}^\rho{}_{\nu\mu}, \quad K^\rho{}_{\mu\nu} = -\frac{1}{2} (T_\nu{}^\rho{}_\mu + T_\mu{}^\rho{}_\nu + T^\rho{}_{\mu\nu}),$$

$$T^\rho{}_{\mu\nu} = -T^\rho{}_{\nu\mu}$$

$$K_{\mu\nu\sigma} = -K_{\sigma\nu\mu}$$

- Assume: connection is not an independent field:

$$\nabla_\rho g_{\mu\nu} = 0 \quad \tilde{\Gamma}^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - K^\rho{}_{\mu\nu} \quad \tilde{\nabla}_\mu V^\nu = \nabla_\mu V^\nu - K^\nu{}_{\mu\lambda} V^\lambda$$

2. Galileons on a spacetime with Torsion.

- The Action with explicit Torsion

We consider a set of 3 independent fields: metric, scalar and contortion with

$$\begin{aligned} \mathcal{S}_{4c} = & \int d^4x \sqrt{-g} \left(G_4(R + K_{\mu\rho\nu}K^{\mu\nu\rho} + K^\mu_{\mu\nu}K^\rho_{\nu\rho} + 2\nabla_\nu K^\mu_{\mu\nu}) + G_{4X}(\nabla_\nu \nabla^\nu \phi + K^{\rho\nu}{}_\nu \nabla_\rho \phi)^2 \right. \\ & - G_{4X}(K^\rho{}_{\gamma\mu} \nabla_\rho \phi + \nabla_\gamma \nabla_\mu \phi)(K^{\nu\gamma\mu} \nabla_\nu \phi + \nabla^\gamma \nabla^\mu \phi) \\ & \left. + (1 - c)G_{4,X}K_{\nu\mu\sigma} (K^{\nu\mu\lambda} - K^{\mu\nu\lambda}) \nabla_\lambda \phi \nabla^\sigma \phi \right). \end{aligned}$$

Why is this Action interesting?

... an apparent kinetic mixing with

Torsion

(In contrast to Einstein-Cartan)

$$G_4 \nabla K$$

$$G_{4,X} (\nabla \phi) (\nabla \nabla \phi) K$$

$$G_4(\phi, X) \quad X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

Some questions arise:

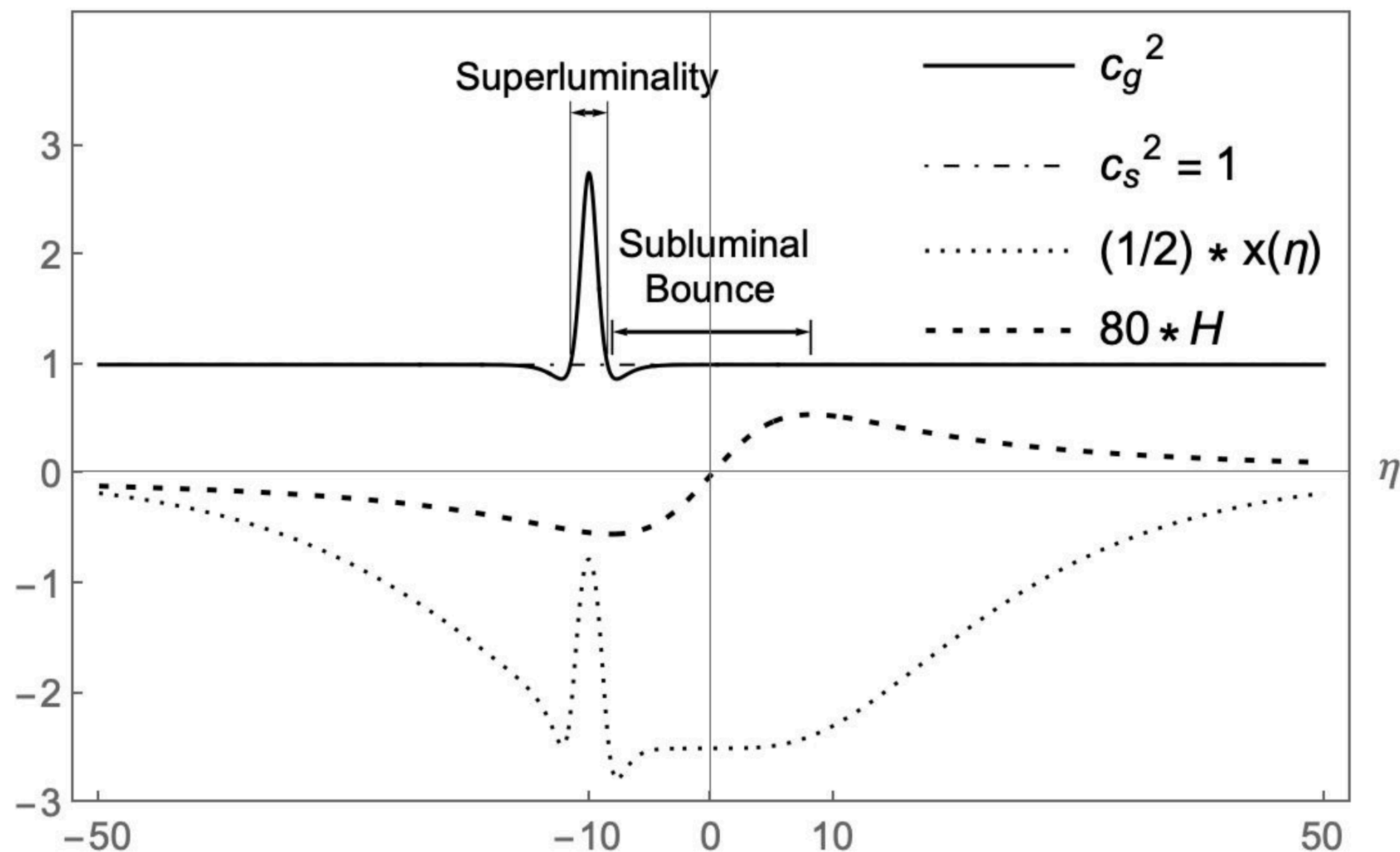
1. Are there more Degrees of Freedom?

2. Different dynamics:

a new chance to stable solutions? Yes

(Recall the No-Go in Torsionless G.)

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3. Torsionful Galileons about the FLRW background

3. Torsionful Galileons about the FLRW background

Address previous questions on a cosmological setting

1. **Are there more Degrees of Freedom?**
2. **Different dynamics: a new chance to stable solutions?**

3. Torsionful Galileons about the FLRW background

- **Linearization: Notation. the perturbed metric**

$$ds^2 = (\eta_{\mu\nu} + \delta g_{\mu\nu}) dx^\mu dx^\nu$$

Spatially flat FLRW background in conformal time

$$\eta_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) (-d\eta^2 + \delta_{ij} dx^i dx^j)$$

4 scalars, 2 (2-component) vectors and a (2-component) tensor
perturbation (graviton)

$$\begin{aligned} \delta g_{\mu\nu} dx^\mu dx^\nu = & a^2(\eta) (-2\alpha d\eta^2 \\ & + 2(\partial_i B + S_i) d\eta dx^i + (-2\psi \delta_{ij} \\ & + 2\partial_i \partial_j E + \partial_i F_j + \partial_j F_i + 2h_{ij}) dx^i dx^j) \end{aligned}$$

3. Torsionful Galileons about the FLRW background

- Linearization: Notation

The perturbed Horndeski scalar: $\phi = \varphi(\eta) + \Pi$

(In the context of linearized expressions we will also denote with ϕ the background)

The perturbed contortion tensor: $K_{\mu\nu\sigma} = {}^0K_{\mu\nu\sigma} + \delta K_{\mu\nu\sigma}$

- For the background contortion: with $K_{\mu\nu\sigma} = -K_{\sigma\nu\mu}$ on an isotropic and homogeneous spacetime

$${}^0K_{0jk} = x(\eta)\delta_{jk}$$

$${}^0K_{ijk} = y(\eta)\epsilon_{ijk}$$

3. Torsionful Galileons about the FLRW background

- Linearization: Notation

- For the perturbation of contortion: with $K_{\mu\nu\sigma} = -K_{\sigma\nu\mu}$,

24 independent components

8 scalars,

$$\delta K_{i00}^{\text{scalar}} = \partial_i C^{(1)}$$

$$\delta K_{ij0}^{\text{scalar}} = \partial_i \partial_j C^{(2)} + \delta_{ij} C^{(3)} + \epsilon_{ijk} \partial_k C^{(4)}$$

$$\delta K_{i0k}^{\text{scalar}} = \epsilon_{ikj} \partial_j C^{(5)}$$

$$\delta K_{ijk}^{\text{scalar}} = (\delta_{ij} \partial_k - \delta_{kj} \partial_i) C^{(6)} + \epsilon_{ikl} \partial_l \partial_j C^{(7)} + (\epsilon_{ijl} \partial_l \partial_k - \epsilon_{kjl} \partial_l \partial_i) C^{(8)}$$

3. Torsionful Galileons about the FLRW background

- Linearization: Notation

6 (2-component) vectors

$$\delta K_{i00}^{\text{vector}} = V_i^{(1)}$$

$$\delta K_{ij0}^{\text{vector}} = \partial_i V_j^{(2)} + \partial_j V_i^{(3)}$$

$$\delta K_{i0k}^{\text{vector}} = \partial_i V_k^{(4)} - \partial_k V_i^{(4)}$$

$$\delta K_{ijk}^{\text{vector}} = \delta_{ij} V_k^{(5)} - \delta_{kj} V_i^{(5)} + \partial_j \partial_i V_k^{(6)} - \partial_j \partial_k V_i^{(6)}$$

and 2 (2-component) tensors

$$\delta K_{ij0}^{\text{tensor}} = T_{ij}^{(1)}$$

$$\delta K_{ijk}^{\text{tensor}} = \partial_i T_{jk}^{(2)} - \partial_k T_{ji}^{(2)}$$

3. Torsionful Galileons about the FLRW background

- **Linearization: structure of the background equations**

- We can solve for H , $\ddot{\phi}(\eta)$ and y , x .

e.g.

$$\mathcal{E}_{K_{ijk}} = \epsilon_{ijk} \frac{2}{a^6} G_4 y = 0, \quad y(\eta) \equiv 0$$

$$\mathcal{E}_{K_{ij0}} = 0, \quad x(\eta) = -\frac{a^3 \mathcal{G}_\tau (8 H X G_{4,X} + a \dot{\phi} (G_3 - 2 G_{4,\phi}))}{8 G_4^2}$$

3. Torsionful Galileons about the FLRW background

Answer to 1st question at linear order about a spatially flat FLRW background

1. Are there more Degrees of Freedom?

Answer: No.

The seeming kinetic terms conspire to cancel out.

Symmetry? Accidental symmetry?

2. ...

3. Torsionful Galileons about the FLRW background

- Final Quadratic Action:

$$S_{4c} = \frac{1}{2} \int d\eta d^3x a^4 \left[\frac{1}{a^2} \left(\mathcal{G}_\tau (\dot{h}_{ij})^2 - \mathcal{F}_\tau (\partial_k h_{ij})^2 \right) + \frac{1}{a^2} \left(\dot{\psi} \left(\mathcal{G}_{SI} - c \frac{1}{a^2} \mathcal{G}_{SII} \partial_i \partial_i \right) \dot{\psi} - \mathcal{F}_S (\partial_i \psi)^2 \right) \right]$$

- One tensor perturbation
- No dynamical vector perturbation
- One scalar perturbation
- **Theory with c=0 is special**
- Can classify theories with NonZero c, based on no-ghost, stability and subluminality conditions.

$$\mathcal{G}_\tau = 2 \frac{G_4^2}{G_4 + 2X G_{4,X}},$$

$$\mathcal{F}_\tau = 2G_4,$$

$$c_g^2 = \mathcal{F}_\tau / \mathcal{G}_\tau$$

4. Stability:

the NO-GO in the Torsionful theory ($c=0$)

4. Stability

Answer to 2nd question in a cosmological setting

2. Different dynamics: a new chance to stable solutions?

Answer: Partially for $c=0$

4. Stability: c=0 theory

$$\mathcal{S}_s = \int d\eta d^3x a^4 \left(\frac{1}{a^2} \mathcal{G}_s \dot{\psi}^2 - \frac{1}{a^2} \mathcal{F}_s (\partial_i \psi)^2 \right)$$

with

$$\mathcal{G}_s = 3 \mathcal{G}_\tau + \frac{\mathcal{G}_\tau^2 \Sigma}{\Theta^2}, \quad \mathcal{F}_s = \frac{1}{a^2} \frac{d}{d\eta} \left(\frac{a \mathcal{G}_\tau T}{\Theta} \right) - \mathcal{F}_\tau$$

$$\mathcal{G}_\tau = 2 \frac{G_4^2}{G_4 + 2X G_{4,X}},$$

$$\mathcal{F}_\tau = 2 G_4,$$

$$T = \mathcal{F}_\tau (c_g^2 - 2)$$

$$c_g^2 = \mathcal{F}_\tau / \mathcal{G}_\tau$$

Key part,

Follow a similar reasoning as initially proved for a subclass of generalized Galileons in (Libanov, Mironov and Rubakov, 2016) and then extended to the full Horndeski action in (Kobayashi, 2016)...

4. The NO-GO in the Torsionful theory ($c=0$)

No-Go for nonsingular, all-time stable and sub/ luminal solutions

For (up to quartic) Galileons on a spacetime with torsion the following assumptions for a first order perturbative expansion about FLRW are mutually inconsistent:

I) *Nonsingular cosmology: namely, there is a lower bound on the scale factor $a(\eta) > b_1 > 0$.*

II) *The graviton and the scalar mode are not ghosts and they suffer no gradient instabilities: $\mathcal{G}_\tau > 0, \mathcal{F}_\tau > 0, \mathcal{F}_S > 0, \mathcal{G}_S > 0$.*

III) **The graviton is always sub/ luminal:** $\boxed{(c_g)^2 \leq 1}$

IV) ...

4. Stability: the NO-GO in the Torsionful theory ($c=0$)

No-Go for nonsingular, all-time stable and sub/ luminal solutions

...

IV) *There is a lower bound $\mathcal{F}_\tau(\eta) > b_2 > 0$ as $\eta \rightarrow \pm\infty$ (no „Strong gravity“ at linear order (Ageeva, Petrov and Rubakov, 2021. See also talk)).*

V) \ominus *Vanishes at most a finite amount of times (To cover generic theories not defined by the equation $\ominus \equiv 0$ (Mironov and Shtennikova, 2023. Also, talk))*

4. Stability: $\mathcal{F}_S > 0$

The argument in Galileons: take $\mathcal{F}_S = \frac{1}{a^2} \frac{d}{d\eta} \left(\frac{a \mathcal{G}_\tau T}{\Theta} \right) - \mathcal{F}_\tau$

With Torsion

/

(without Torsion)

$$T = \mathcal{F}_\tau (c_g^2 - 2)$$

$$T = \mathcal{G}_\tau$$

Argument With Torsion: (I)-(III) imply

$$N =: \frac{a \mathcal{G}_\tau \mathcal{F}_\tau (c_g^2 - 2)}{\Theta} \neq 0,$$

Because Θ is a regular function of H and ϕ

4. Stability:

$$\mathcal{F}_S = \frac{1}{a^2} \frac{d}{d\eta} \left(\frac{a \mathcal{G}_\tau T}{\Theta} \right) - \mathcal{F}_\tau$$

Now integrate $\mathcal{F}_S > 0$

$$\Delta N = N_f - N_i > I(\eta_i, \eta_f),$$
$$I(\eta_i, \eta_f) = \int_{\eta_i}^{\eta_f} d\eta a^2 \mathcal{F}_\tau,$$

with N_f and N_i the values of N at some (conformal) times η_f and η_i respectively

4. Stability

A) $a^2 \mathcal{F}_\tau > 0$

$$I(\eta_i) = I(\eta_i, \eta_f)|_{\eta_f} \quad \text{positive, growing with } \eta_i$$

$$I(\eta_f) = I(\eta_i, \eta_f)|_{\eta_i} \quad \text{positive, growing with } \eta_f$$

B) $\Delta N > 0$

C) $I(\eta_i)$ not convergent as $\eta_i \rightarrow -\infty$

$I(\eta_f)$ not convergent as $\eta_f \rightarrow \infty$

4. Stability

Now, take

$$-\infty < N_i < 0$$

Since $N \neq 0$, follows

$$N_f(\eta_f) < 0$$

and

$$|N_i| > \Delta N = |N_i| - |N_f|$$

so, there exist η_c such that

$$\boxed{\text{if } \eta_f > \eta_c, I(\eta_f) > |N_i| > \Delta N}$$

namely,

$$\Delta N \not\geq I(\eta_i, \eta_f)$$

4. Stability

Similarly, take

$$\infty > N_f > 0$$

Since $N \neq 0$, follows

$$N_i(\eta_i) > 0$$

and

$$N_f > \Delta N = N_f - N_i$$

so, there exist η_c such that

$$\boxed{\text{if } \eta_i < \eta_c, I(\eta_i) > N_f > \Delta N}$$

namely,

$$\Delta N \not\geq I(\eta_i, \eta_f)$$

4. Stability

Namely,

$$\Delta N \not\geq I(\eta_i, \eta_f)$$

And

$$\boxed{\mathcal{F}_S \not\geq 0}$$

**No-Go for nonsingular, all-time stable and sub/ luminal
solutions**

(In torsionful G. stability -sub/luminality)

4. **Stability:** *example.* A proof of principle

4. Stability: a proof of principle, recovering GR

By-pass the no-go? **short-lived superluminality**

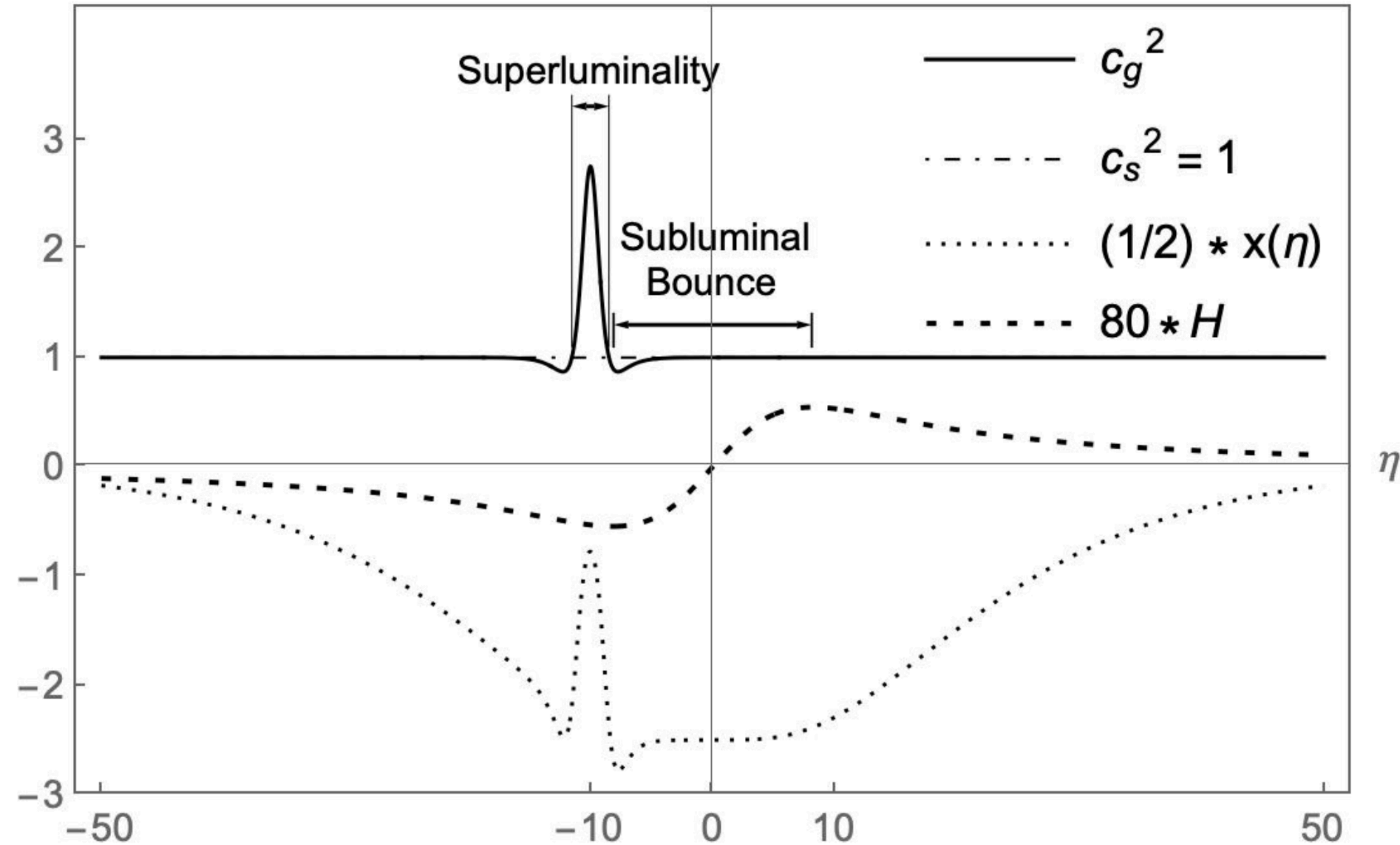
$$c_g \geq \sqrt{2} c$$

Minimal example: non zero G4X

(Reconstruct S)

$$\eta_s < \eta_b = 0$$

$$\tau_s \ll \tau_b$$



A model with an all-time stable non-singular cosmology with a short period of superluminality of the graviton. S. Mironov and M. Valencia-Villegas, 2307.06929

$$S^\infty = \frac{1}{2} \int d^4x \sqrt{-g} (R - \partial_\mu \xi \partial^\mu \xi)$$

$$\xi = \sqrt{\frac{3}{2}} \ln(\phi) \quad x = \mp \eta e^{\mp \frac{\eta}{\tau_b}}$$

Conclusions

- No extra DOFs. Classification of the scalar according to the parameter c of the theory.

S. Mironov and M. Valencia-Villegas, Phys. Rev. D, vol. 108, no. 2, p. 024057, 2023

- *In generic models of (up to quartic) Galileons with torsion it is not possible to obtain a nonsingular FLRW cosmology that is always free of gradient instabilities against the scalar perturbation and an eternally sub/ luminal graviton.*

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- A spacetime with torsion can support all-time linearly stable nonsingular solutions in Galileons if there exists at an arbitrary time a superluminal phase for the graviton and by at least an amount $c_g \geq \sqrt{2}c$
- The unphysical, superluminal phase can formally happen as a deep UV inconsistency (arbitrarily short) and unrelated to the physically relevant scale e.g. width of a bounce.
- Horndeski-Cartan theory is different to Horndeski on a torsionless geometry, in contrast to e.g. the equivalence of Einstein-Cartan and GR without matter. (Stability-Sub/Luminality)

Open questions:

- Accidental symmetry [20 - 23]?
- Lorentz invariant UV completions for models with all-time stable nonsingular cosmologies (Adams *et.al.* , 2006), (Dubovsky *et.al.* , 2006)?
- G5 changes the picture?
 - Indeed...

4. Stability: reconstruct S

Out of G2, G3, G4 reconstruct a Lagrangian for the fixed solutions
(Inverse method, see also [19])

$$a = (\tau_b^2 + \eta^2)^{\frac{1}{4}}, \quad H = \frac{\dot{a}}{a^2} = \frac{\eta}{2(\tau_b^2 + \eta^2)^{\frac{5}{4}}}, \quad \phi = \eta,$$

Thus, in linearized expressions

$$X = \frac{\dot{\phi}^2}{2a^2}$$

$$X = 1/(2(\tau_b^2 + \eta^2)^{\frac{1}{2}})$$

4. Stability: example

Demand GR asymptotics as $\eta \rightarrow \pm\infty$

$$G_2(\phi, X) \rightarrow \frac{1}{2a^2} \dot{\xi}^2, \quad G_4(\phi, X) \rightarrow \frac{1}{2}, \quad M_{pl}^2/8\pi = 1$$

$$G_3(\phi, X) \rightarrow 0, \quad x(\eta) \rightarrow 0.$$

ξ is some invertible function of the Horndeski scalar.

And we assume $\eta_s < \eta_b = 0$

4. Stability: example

The following Ansatz for the model has enough structure

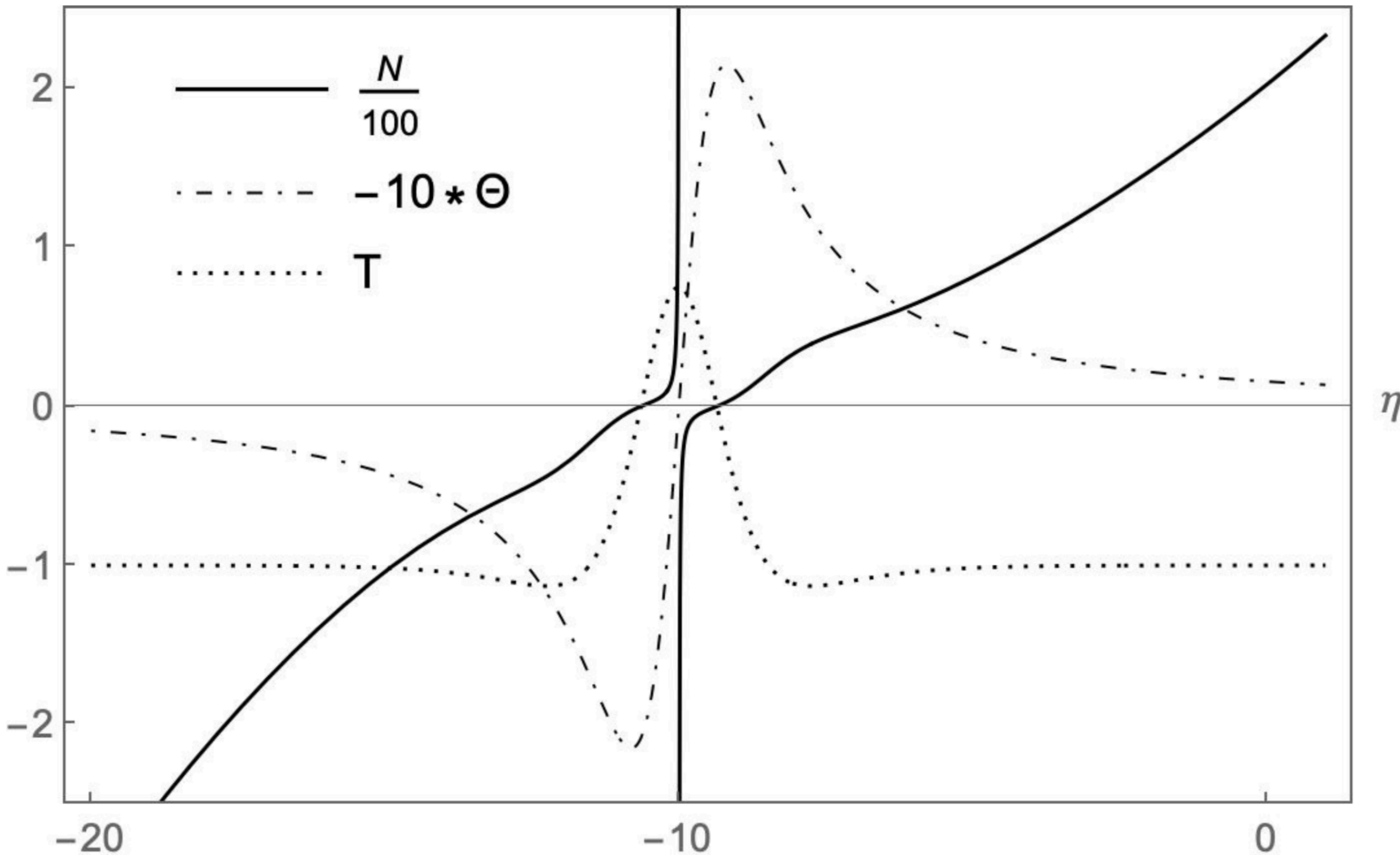
$$G_2(\phi, X) = g_{20}(\phi) + g_{21}(\phi) X + g_{22}(\phi) X^2 ,$$

$$G_3(\phi, X) = g_{30}(\phi) + g_{31}(\phi) X ,$$

$$G_4(\phi, X) = \frac{1}{2} + g_{40}(\phi) + g_{41}(\phi) X .$$

4. Stability: example

$$\mathcal{G}_\tau > 0, \mathcal{F}_\tau > 0, \mathcal{F}_S > 0, \mathcal{G}_S > 0$$



$$T = \mathcal{F}_\tau (c_g^2 - 2)$$

$$N =: \frac{a \mathcal{G}_\tau \mathcal{F}_\tau (c_g^2 - 2)}{\Theta} \neq 0,$$

4. Stability: example

The solutions for the Lagrangian functions take the following form

as $\eta \rightarrow \pm\infty$

$$g_{40} = -g_{41} X = \frac{5}{8} e^{\mp \frac{(\eta - \eta_s)}{\tau_s}}$$

$$g_{30} = -g_{31} X = \frac{3}{2} \frac{\eta_s}{\eta^2} e^{\mp \frac{\tau_s}{\tau_b} \frac{(\eta - \eta_s)}{|\eta_s|}}$$

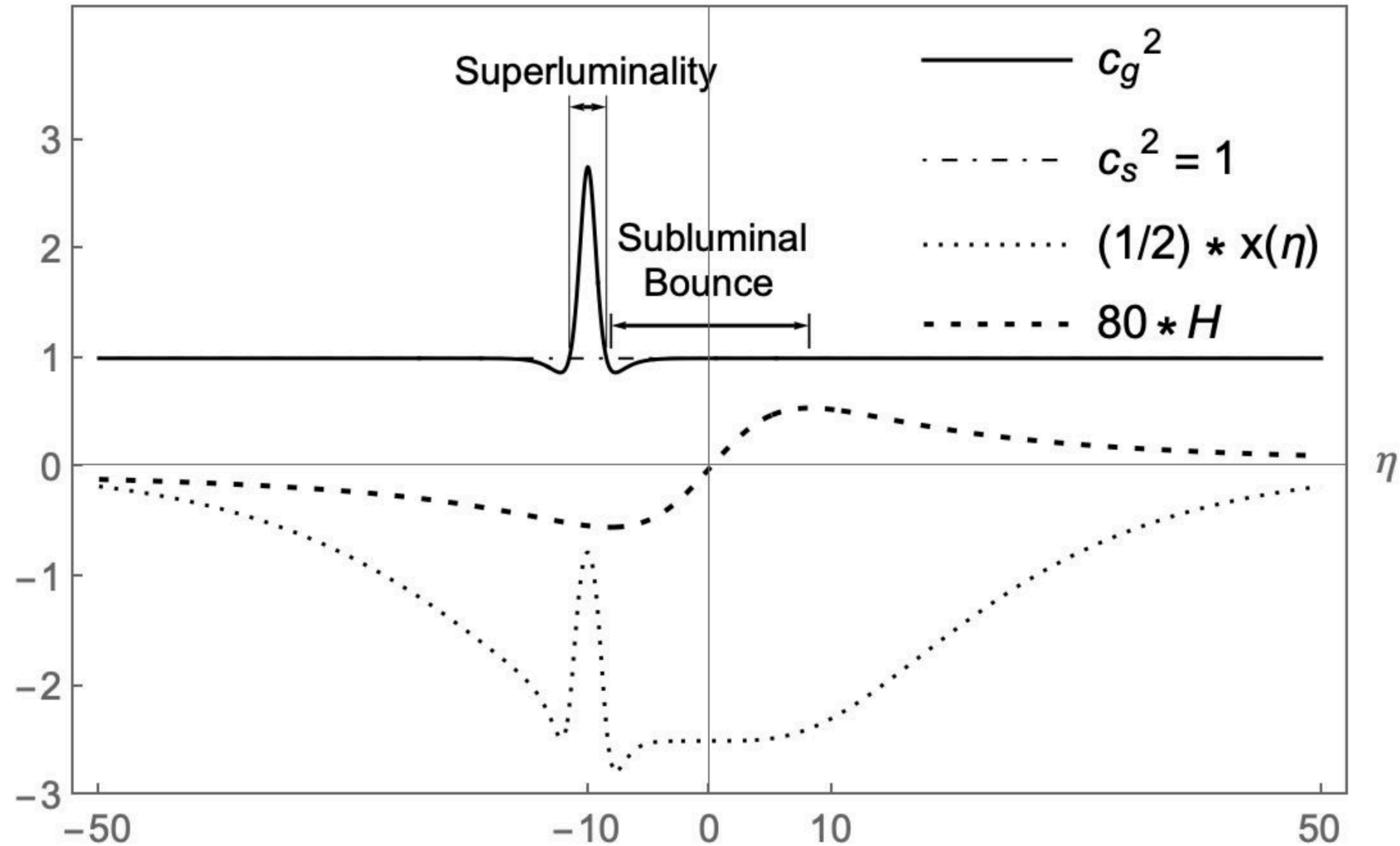
$$x = \mp \eta e^{\mp \frac{\eta}{\tau_b}}$$

$$g_{20} = -\frac{\tau_b^2}{2} (\pm\eta)^{-5}, \quad g_{21} X = \frac{3}{4} (\pm\eta)^{-3}$$

$$g_{22} X^2 = \mp \frac{3}{4} (\pm\eta)^{-3} \frac{\tau_s}{\tau_b} e^{\mp \frac{\tau_s}{\tau_b} \frac{(\eta - \eta_s)}{|\eta_s|}}$$

4. Stability: example

$$\mathcal{G}_\tau > 0, \mathcal{F}_\tau > 0, \mathcal{F}_S > 0, \mathcal{G}_S > 0$$



4. Stability: example

Asymptotic Lagrangian

The leading expressions in the Ansatz

$$G_2(\phi, X) = g_{20}(\phi) + g_{21}(\phi) X + g_{22}(\phi) X^2,$$

$$G_3(\phi, X) = g_{30}(\phi) + g_{31}(\phi) X,$$

$$G_4(\phi, X) = \frac{1}{2} + g_{40}(\phi) + g_{41}(\phi) X.$$

as $\eta \rightarrow \pm\infty$

Are $G_4 = \frac{1}{2}$, $g_{20} = -\frac{\tau_b^2}{2} (\pm\eta)^{-5}$, $g_{21} X = \frac{3}{4} (\pm\eta)^{-3}$

Now, with the leading solutions $a = \eta^{\frac{1}{2}}$, $H = \frac{1}{2}\eta^{-\frac{3}{2}}$, $\phi = \eta$, $X = 1/(2\eta)$

The corresponding action to S in the asymptotic past and future is

$$S^\infty = \frac{1}{2} \int d^4x \sqrt{-g} (R - \partial_\mu \xi \partial^\mu \xi) \quad \xi = \sqrt{\frac{3}{2}} \ln(\phi)$$

Indeed, the leading solutions satisfy $\ddot{\xi} + 2aH\dot{\xi} = 0$, $\dot{\xi}^2 - 6a^2H^2 = 0$

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Additional Material

Classification of the scalar according to the parameter c of the theory

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Table 1. Classification of the scalar according to the parameter c of the theory.

	$c < 0$	$c = 0$	$0 < c \leq 2$	$c > 2$
Scalar mode	Non wave-like dispersion relation. <i>Not a ghost</i> (in high momentum) if the graviton is healthy*.	<i>Wave-like</i> dispersion relation.	Non wave-like dispersion relation. <i>A ghost</i> (in high momentum) if the graviton is healthy*.	Non wave-like dispersion relation.
Graviton	Is massless. The no ghost, stability and subluminality conditions ($\mathcal{G}_\tau > 0$, $\mathcal{F}_\tau > 0$, $\frac{\mathcal{F}_\tau}{\mathcal{G}_\tau} < 1$) are satisfied if $G_4 > -2 X G_{4,X} > 0$.			
Vector sector	Non dynamical.			

S. Mironov and M. Valencia-Villegas, Phys. Rev. D, vol. 108, no. 2, p. 024057, 2023

$$S_{4c} = \frac{1}{2} \int d\eta d^3x a^4 \left[\frac{1}{a^2} \left(\mathcal{G}_\tau (\dot{h}_{ij})^2 - \mathcal{F}_\tau (\partial_k h_{ij})^2 \right) + \frac{1}{a^2} \left(\dot{\psi} \left(\mathcal{G}_{SI} - c \frac{1}{a^2} \mathcal{G}_{SII} \partial_i \partial_i \right) \dot{\psi} - \mathcal{F}_S (\partial_i \psi)^2 \right) \right]$$

$$c \mathcal{G}_{SII} = \frac{8 c G_{4,X}^3 G_4^3}{(G_4 + c X G_{4,X}) (G_{4,X} G_{4,\phi} - G_4 G_{4,\phi X})^2} > 0$$

Additional Material

Resolving ambiguities in the definition of Galileons with torsion

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- Resolving ambiguities in the definition of Galileons with torsion

Recall **Torsionless G4**: $\mathcal{S}_4 = \int d^4x \sqrt{-g} \left(G_4(\phi, X) R + G_{4,X} \left((\nabla_\mu \nabla^\mu \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right) \right)$

Here, the metric compatible derivative $\nabla_\rho g_{\mu\nu} = 0$ on a vector V , is

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda, \quad \Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

such that $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$ and $[\nabla_\mu, \nabla_\nu] \phi = 0$.

-> There is no ambiguity in $G_{4,X} (\nabla_\mu \nabla_\nu \phi)^2$

Additional Material

- Resolving ambiguities in the definition of Galileons with torsion

To go to **Torsionful G4** take $\nabla \rightarrow \tilde{\nabla}$, but then, what is $(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi)^2$?

Here, the metric compatible derivative $\nabla_\rho g_{\mu\nu} = 0$ on a vector is

$$\tilde{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}_{\mu\lambda}^\nu V^\lambda, \quad \tilde{\Gamma}_{\mu\lambda}^\nu \neq \tilde{\Gamma}_{\lambda\mu}^\nu$$

so, $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] \phi \neq 0$ -> there are two possible contractions with the metric for $G_{4,X}$ $(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi)^2$, namely

$$G_{4,X} \left(\left(\tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi \right)^2 + c \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right) \tilde{\nabla}^\mu \tilde{\nabla}^\nu \phi + s \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right) \tilde{\nabla}^\nu \tilde{\nabla}^\mu \phi \right), \quad c + s = -1$$

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] \phi = -T^\lambda_{\mu\nu} \partial_\lambda \phi$$

Additional Material

3 Second order field equations

Additional Material

S3 Second order field equations:

Hence the field equations look like

$$\mathcal{E}_\phi(\tilde{\nabla}^2 K, \tilde{\nabla}^2 \phi, \partial^2 g) = 0, \quad \mathcal{E}_{g_{\mu\nu}}(\tilde{\nabla}^2 \phi, \partial^2 g) = 0, \quad \mathcal{E}_{K^{\mu\nu\sigma}}(\tilde{\nabla}^2 \phi) = 0,$$

e.g.

$$\mathcal{E}_\phi = 2 G_{4X} \partial^\lambda \phi \left(\tilde{\nabla}_\lambda \tilde{\nabla}_\mu K^\nu{}_\nu{}^\mu - \tilde{\nabla}_\nu \tilde{\nabla}^\nu K^\mu{}_\mu{}_\lambda + \tilde{\nabla}_\nu \tilde{\nabla}_\mu K^{\nu\mu}{}_\lambda \right) + F(K, \tilde{\nabla} K; \tilde{\nabla}^2 \phi, \tilde{\nabla} \phi, \tilde{R})$$

Additional Material

Quadratic Action

3. Torsionful Galileons about the FLRW background

- **Quadratic Action:** $\mathcal{S}_{4c} = \mathcal{S}^{Tensor} + \mathcal{S}_c^{Scalar}$

$$\mathcal{S}^{Tensor} = \frac{1}{2} \int d\eta d^3x \left(v_1 (\dot{h}_{ij})^2 + v_2 (\partial_k h_{ij})^2 + v_3 (T_{ij}^{(1)})^2 + v_4 (\partial_k T_{ij}^{(2)})^2 + v_5 h_{ij} T_{ij}^{(1)} + v_6 \dot{h}_{ij} T_{ij}^{(1)} + v_7 (h_{ij})^2 \right)$$

$$T_{ij}^{(1)} = \frac{2 a^2 X G_{4,X}}{G_4 + 2 X G_{4,X}} \dot{h}_{ij} - 2 x h_{ij} \quad T_{ij}^{(2)} \equiv 0$$

$$\mathcal{S}_\tau = \int d\eta d^3x a^4 \left[\frac{1}{2 a^2} \left(\mathcal{G}_\tau (\dot{h}_{ij})^2 - \mathcal{F}_\tau (\partial_k h_{ij})^2 \right) \right]$$

$$\mathcal{G}_\tau = 2 \frac{G_4^2}{G_4 + 2 X G_{4,X}},$$

$$c_g^2 = \mathcal{F}_\tau / \mathcal{G}_\tau$$

$$X = \frac{\dot{\phi}^2}{2 a^2}$$

$$\mathcal{F}_\tau = 2 G_4,$$

Additional Material

Scalar sector

$$\begin{aligned} \mathcal{S}_c^{Scalar} = & \frac{1}{2} \int d\eta d^3x \left(c \left(f_7 \partial_i B \partial_i C^{(1)} + f_{51} (\partial_i B)^2 + f_{52} (\partial_i C^{(1)})^2 \right) \right. \\ & + \left(f_1 \alpha \Pi + f_2 C^{(3)} \psi + f_3 \alpha \psi + f_4 \Pi \psi + f_5 C^{(3)} \Pi + f_6 C^{(3)} \alpha + f_8 \partial_i C^{(3)} \partial_i E \right. \\ & + f_9 \partial_i B \partial_i C^{(3)} + f_{10} \partial_i C^{(2)} \partial_i C^{(3)} + f_{11} \partial_i C^{(4)} \partial_i C^{(5)} + f_{12} \partial_i B \partial_i C^{(6)} + f_{13} \partial_i C^{(1)} \partial_i C^{(6)} \\ & + f_{14} \partial_i B \partial_i \Pi + f_{15} \partial_i E \partial_i \Pi + f_{16} \partial_i C^{(1)} \partial_i \Pi + f_{17} \partial_i C^{(2)} \partial_i \Pi + f_{18} \partial_i C^{(3)} \partial_i \Pi + f_{19} \partial_i C^{(6)} \partial_i \Pi \\ & + f_{20} \partial_i B \partial_i \alpha + f_{21} \partial_i E \partial_i \alpha + f_{22} \partial_i C^{(2)} \partial_i \alpha + f_{23} \partial_i C^{(6)} \partial_i \alpha + f_{24} \partial_i \alpha \partial_i \Pi + f_{25} \partial_i E \partial_i \psi \\ & + f_{26} \partial_i B \partial_i \psi + f_{27} \partial_i C^{(2)} \partial_i \psi + f_{28} \partial_i \Pi \partial_i \psi + f_{29} \partial_i \alpha \partial_i \psi + f_{30} \partial_i \partial_j C^{(7)} \partial_i \partial_j C^{(8)} + f_{31} \psi \dot{\Pi} \\ & + f_{32} \alpha \dot{\Pi} + f_{33} C^{(3)} \dot{\Pi} + f_{34} \alpha \dot{\psi} + f_{35} C^{(3)} \dot{\psi} + f_{36} \dot{\Pi} \dot{\psi} + f_{37} \partial_i C^{(3)} \partial_i \dot{E} + f_{38} \partial_i \alpha \partial_i \dot{E} \\ & + f_{39} \partial_i B \partial_i \dot{\Pi} + f_{40} \partial_i E \partial_i \dot{\Pi} + f_{41} \partial_i C^{(2)} \partial_i \dot{\Pi} + f_{42} \partial_i C^{(6)} \partial_i \dot{\Pi} + f_{43} \partial_i \dot{E} \partial_i \dot{\Pi} + f_{44} \partial_i B \partial_i \dot{\psi} \\ & + f_{45} \partial_i C^{(2)} \partial_i \dot{\psi} + f_{46} \partial_i \dot{E} \partial_i \dot{\psi} + f_{47} (C^{(3)})^2 + f_{48} \alpha^2 + f_{49} \psi^2 + f_{50} \Pi^2 + f_{53} (\partial_i C^{(4)})^2 \\ & \left. \left. + f_{54} (\partial_i C^{(6)})^2 + f_{55} (\partial_i \Pi)^2 + f_{56} (\partial_i \psi)^2 + f_{57} (\partial_i \partial_j C^{(8)})^2 + f_{58} \dot{\Pi}^2 + f_{59} \dot{\psi}^2 \right) \right), \end{aligned}$$

Additional Material

Scalar sector:

How do we get $\boxed{\mathcal{G}_\tau \neq T}$ in the torsionful Galileons?

Answer: a nontrivial torsion scalar coupled to the dynamical scalar mode

$$C^{(3)} = -\frac{2a^2 X G_{4,X}}{G_4 + 2X G_{4,X}} \dot{\psi} + 2x\psi - \frac{a^3 (2G_4 H + \Theta) + a^2 \dot{\phi} G_{4,\phi}}{2G_4} \alpha$$

Additional Material

Details for theory with $c=0$

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(G_2 - G_3 \tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi + G_4 \tilde{R} \right. \\ \left. + G_{4,X} \left(\left(\tilde{\nabla}_\mu \tilde{\nabla}^\mu \phi \right)^2 - \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right) \tilde{\nabla}^\nu \tilde{\nabla}^\mu \phi \right) \right)$$

Scalar sector very similar to torsionless Galileons, **Except** $\boxed{\mathcal{G}_\tau \neq T}$

$$\mathcal{S}_s = \int d\eta d^3x a^4 \left(-3 \frac{\mathcal{G}_\tau}{a^2} \dot{\psi}^2 \right. \\ \left. + \frac{\mathcal{F}_\tau}{a^2} (\partial_i \psi)^2 + 6 \frac{\Theta}{a} \alpha \dot{\psi} + 2 \frac{T}{a^2} \partial_i \alpha \partial_i \psi \right. \\ \left. + 2 \frac{\partial_i \partial_i B}{a^2} \left(a \Theta \alpha - \mathcal{G}_\tau \dot{\psi} \right) + \Sigma \alpha^2 \right)$$

$$\mathcal{G}_\tau = 2 \frac{G_4^2}{G_4 + 2X G_{4,X}}, \quad \mathcal{F}_\tau = 2G_4, \quad T = \mathcal{F}_\tau (c_g^2 - 2)$$

$$\Theta = \frac{4\mathcal{G}_\tau^2 \theta}{a \mathcal{F}_\tau^4}, \quad \Sigma = \frac{2\mathcal{G}_\tau^3 \sigma}{a \mathcal{F}_\tau^6},$$

$$\alpha = \frac{1}{a} \frac{\mathcal{G}_\tau}{\Theta} \dot{\psi}$$

Additional Material

Example

Additional Material

Indeed, g_{20} , g_{21} , g_{22} , g_{30} , g_{31} , g_{40} and g_{41} can be solved algebraically from the following 7 equations

$$\mathcal{F}_\tau(g_{40}, g_{41}) = 1,$$

$$T(g_{40}, g_{41}) = -1 - \frac{5}{4} \operatorname{Sech} \left(\frac{\eta - \eta_s}{\tau_s} \right) + 3 \operatorname{Sech} \left(\frac{\eta - \eta_s}{\tau_s} \right)^2$$

$$G_3(g_{30}, g_{31}) = \operatorname{Sech} \left(\frac{\eta}{\tau_b} \right)$$

$$\boxed{\mathcal{F}_S > 0}$$

$$\Theta(g_{30}, g_{31}) = -H_s$$

$$\mathcal{G}_S = \mathcal{F}_S$$

$$H_s = \frac{\eta - \eta_s S}{2 (\tau_b^2 (1 - S) + \tau_s^2 S + (\eta - \eta_s S)^2)^{\frac{5}{4}}}$$

$$\mathcal{E}_{g_{00}} = 0$$

$$\mathcal{E}_{g_{33}} = 0$$

$$S = \operatorname{Sech} \left(\frac{\tau_s (\eta - \eta_s)}{\tau_b \eta_s} \right) \quad \Theta \xrightarrow{\eta \rightarrow \pm\infty} -H$$

Additional Material

Plot of the analytical solutions for g_{20} , g_{21} , g_{22} , g_{30} , g_{31} , g_{40} and g_{41}

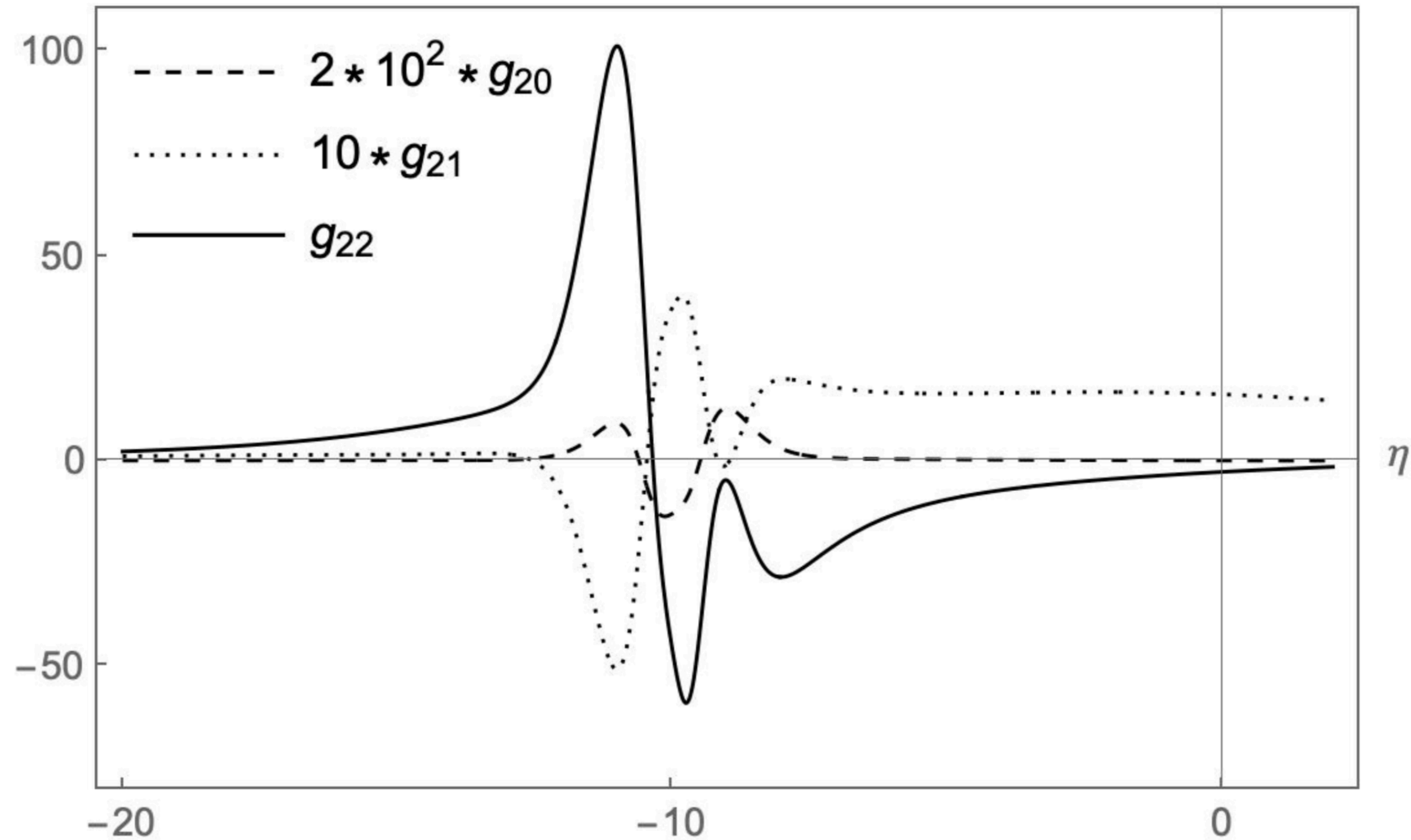
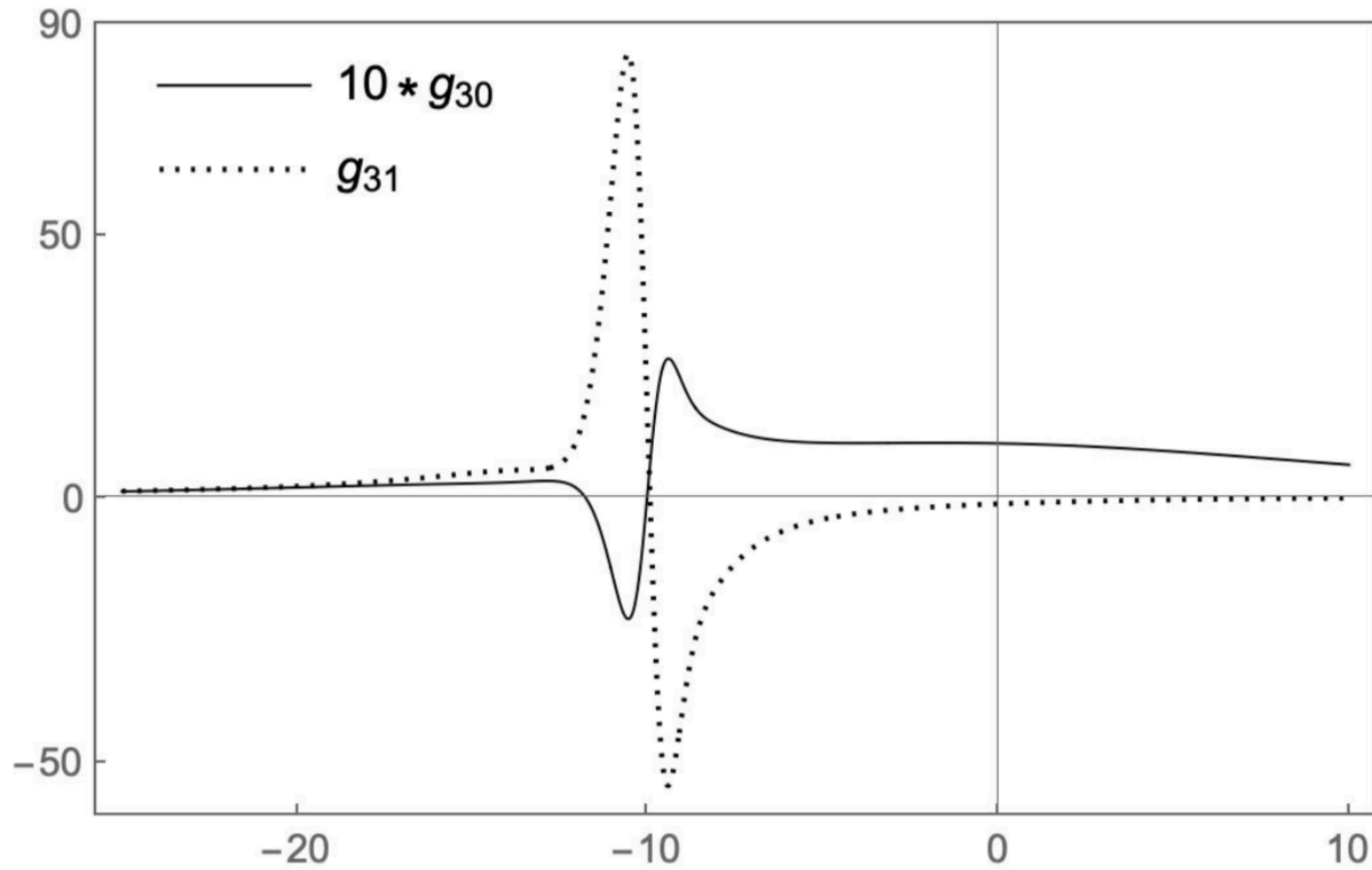


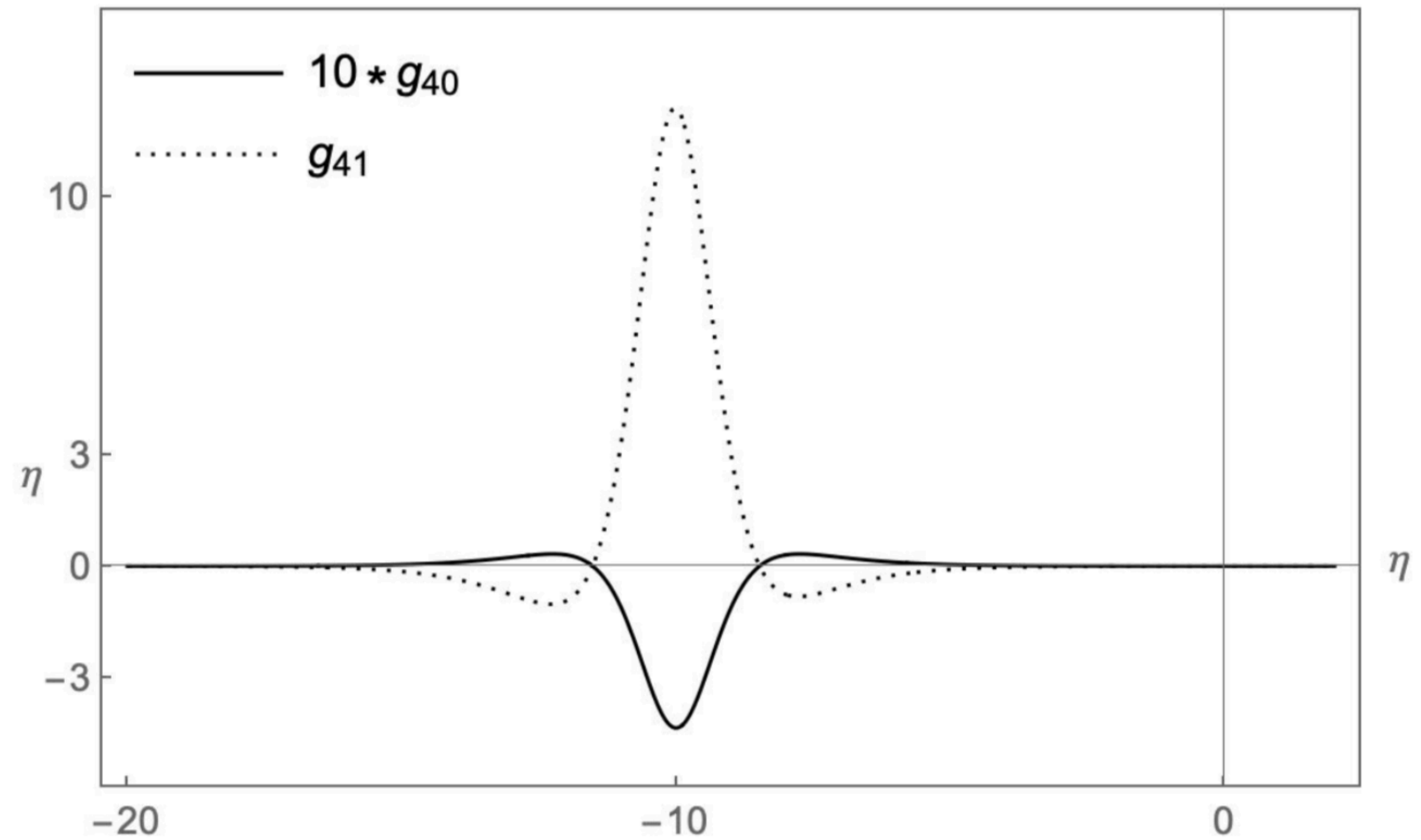
FIG. 3: Everywhere regular Lagrangian functions g_{20} , g_{21} and g_{22} .

Additional Material

Plot of the analytical solutions for g_{20} , g_{21} , g_{22} , g_{30} , g_{31} , g_{40} and g_{41}



(a) g_{30} and g_{31} .



(b) g_{40} and g_{41} .

FIG. 4: Everywhere regular Lagrangian functions.

Additional Material

Non wave like dispersion relation

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Non wave like dispersion relation

$$\mathcal{L}_{toy} = (1 - cp^2) \dot{x}_1^2 - p^2 x_1^2,$$

equivalent

$$\mathcal{L}_{toy'} = \dot{x}_1^2 - p^2 x_1^2 + cp^2 (2x_2 \dot{x}_1 + x_2^2) \quad (x_2 = -\dot{x}_1)$$

Key terms in action $\dot{\psi}^2, p^2 \psi^2, p^2 (C^{(6)})^2$

$$p^2 f_{23} C^{(6)} \alpha \quad \alpha = \dot{\psi} + \dots$$

$$C^{(6)} = -c \frac{1}{f_{13}} (2 f_{52} C^{(1)} + f_7 B)$$

Additional Material

Classification of the scalar for nonzero c

Additional Material

Classification of the scalar for nonzero c

In high momentum

$$c\mathcal{G}_{SII} = \frac{8cG_{4,X}^3 G_4^3}{(G_4 + cX G_{4,X})(G_{4,X} G_{4,\phi} - G_4 G_{4,\phi X})^2} > 0 \quad \mathcal{F}_S > 0$$

From tensor sector $\mathcal{G}_\tau > 0$, $\mathcal{F}_\tau > 0$, and sub/ luminality $G_4 > -2X G_{4,X} > 0$

$$X = \frac{1}{2a^2} \dot{\varphi}^2 > 0 \quad G_4 > 0, \quad G_{4,X} < 0,$$

then

$$c\mathcal{G}_{SII} > 0 \quad \frac{c}{(G_4 + cX G_{4,X})} < 0$$

Rewrite

$$G_4 > -2X G_{4,X} > 0 \quad (G_4 + cX G_{4,X}) > (c - 2)X G_{4,X}$$

Additional Material

Background equations

Additional Material

Background equations $\mathcal{E}_\varphi = 0$, $\mathcal{E}_{g_{00}} = 0$, $\mathcal{E}_{g_{ij}} = 0$, $\mathcal{E}_{K_{ij0}} = 0$,

$$\mathcal{E}_{g_{00}} = (x + a\dot{a}) \left(\frac{3G_4(x - a\dot{a})}{a^8} - \frac{3G_{4,\phi}\dot{\varphi}}{a^6} + \frac{6G_{4,X}(2x + a\dot{a})\dot{\varphi}^2}{a^{10}} - \frac{3G_{4,\phi X}\dot{\varphi}^3}{a^8} + \frac{3G_{4,XX}(x + a\dot{a})\dot{\varphi}^4}{a^{12}} \right),$$

$$\mathcal{E}_{g_{ij}} = \delta_{ij} \left(\frac{G_4\left(-x^2 + a^4\left(\frac{\dot{a}^2}{a^2} + 2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\right)\right)}{a^8} + \frac{G_{4,\phi\phi}\dot{\varphi}^2}{a^4} + \frac{2G_{4,XX}(x + a\dot{a})\dot{\varphi}^3\left(-\ddot{\varphi} + \frac{\dot{a}\dot{\varphi}}{a}\right)}{a^{10}} \right. \\ \left. + \frac{G_{4,\phi X}\dot{\varphi}^2\left(-2x\dot{\varphi} + a^2\left(\ddot{\varphi} - \frac{3\dot{a}\dot{\varphi}}{a}\right)\right)}{a^8} + \frac{G_{4,\phi}\left(-x\dot{\varphi} + a^2\left(\ddot{\varphi} + \frac{\dot{a}\dot{\varphi}}{a}\right)\right)}{a^6} \right. \\ \left. + \frac{G_{4,X}\dot{\varphi}\left(-x^2\dot{\varphi} + a^4\left(-\frac{2\ddot{\varphi}\dot{a}}{a} + \frac{\dot{a}^2\dot{\varphi}}{a^2} - 2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\dot{\varphi}\right) + a^2\left(-4x\ddot{\varphi} + \frac{6x\dot{a}\dot{\varphi}}{a} - 2\dot{\varphi}\dot{x}\right)\right)}{a^{10}} \right),$$

$$\mathcal{E}_{K_{ij0}} = \delta_{ij} \left(\frac{2G_4x}{a^6} - \frac{G_{4,\phi}\dot{\varphi}}{a^4} + \frac{2G_{4,X}(x + a\dot{a})\dot{\varphi}^2}{a^8} \right),$$

$$\mathcal{E}_\varphi = -\frac{2a^2}{\dot{\varphi}^2} \left(\dot{\mathcal{E}}_{g_{00}} + (5\mathcal{E}_{g_{00}} + 3\mathcal{E}_{g_{ii}}) \frac{\dot{a}}{a} \right) + \frac{6x}{\dot{\varphi}^2} \left(\dot{\mathcal{E}}_{K_{ii0}} + 4\mathcal{E}_{K_{ii0}} \frac{\dot{a}}{a} \right),$$

Additional Material

Gauge transformations

Additional Material

Gauge transformations

$$\alpha \rightarrow \alpha - \dot{\xi}^0 - \frac{\dot{a}}{a} \xi^0 \quad B \rightarrow B + \xi^0 - \dot{\xi} \quad \psi \rightarrow \psi + \frac{\dot{a}}{a} \xi^0 \quad E \rightarrow E - \xi$$

$$S_i \rightarrow S_i - \dot{\xi}_i \quad F_i \rightarrow F_i - \xi_i \quad h_{ij} \rightarrow h_{ij},$$

$$\Pi \rightarrow \Pi - \xi^0 \dot{\varphi},$$

$$T_{ij}^{(1)} \rightarrow T_{ij}^{(1)} \quad T_{ij}^{(2)} \rightarrow T_{ij}^{(2)} \quad V_i^{(5)} \rightarrow V_i^{(5)} \quad V_i^{(6)} \rightarrow V_i^{(6)}$$

$$V_i^{(1)} \rightarrow V_i^{(1)} + \dot{\xi}_i x \quad V_i^{(2)} \rightarrow V_i^{(2)} + \xi_i x - \dot{\omega}_i y \quad V_i^{(3)} \rightarrow V_i^{(3)} + \xi_i x + \dot{\omega}_i y \quad V_i^{(4)} \rightarrow V_i^{(4)} + \dot{\omega}_i y$$

$$C^{(1)} \rightarrow C^{(1)} + \dot{\xi} x \quad C^{(2)} \rightarrow C^{(2)} + 2\xi x \quad C^{(3)} \rightarrow C^{(3)} + \xi^0 \dot{x} + \dot{\xi}^0 x \quad C^{(4)} \rightarrow C^{(4)} - \dot{\xi} y$$

$$C^{(5)} \rightarrow C^{(5)} + \dot{\xi} y \quad C^{(6)} \rightarrow C^{(6)} + \xi^0 x \quad C^{(7)} \rightarrow C^{(7)} + \xi y - \frac{1}{|\vec{p}|^2} \dot{y} \xi^0 \quad C^{(8)} \rightarrow C^{(8)} - \xi y + \frac{1}{|\vec{p}|^2} \dot{y} \xi^0,$$

$$\xi_k = \epsilon_{ijk} \partial_i \omega_j$$

$$\partial_i \omega_i = 0$$

Additional Material

Coefficients in scalar sector ($c=0$)

Additional Material

Coefficients in scalar sector (c=0)

$$\theta = -2a G_4 H(G_4^2 + 4(2G_{4,X}^2 - G_{4,XX} G_4) X^2) - (G_{4,\phi}(G_4^2 + 4G_{4,X}^2 X^2 - 2G_4 X(G_{4,X} + 2G_{4,XX} X)) + G_4 X(-(G_{3,X} - 2G_{4,\phi X})(G_4 + 2G_{4,X} X) + G_3(3G_{4,X} + 2G_{4,XX} X))) \dot{\phi},$$

$$\begin{aligned} \sigma = & a(-24H^2(G_4^4 + 16G_{4,X}^4 X^4 + 8G_{4,X}^2 G_4 X^3(G_{4,X} - 2G_{4,XX} X) + 2G_4^2 X^2(15G_{4,X}^2 + 8G_{4,XX}^2 X^2 \\ & + 4G_{4,X} X(G_{4,XX} - G_{4,XXX} X)) - G_4^3 X(G_{4,X} + 4X(4G_{4,XX} + G_{4,XXX} X))) \\ & + X(-3G_3^2(G_4^2 + 2X^2(5G_{4,X}^2 + 8G_{4,XX}^2 X^2 + 4G_{4,X} X(G_{4,XX} - G_{4,XXX} X)) - G_4 X(11G_{4,X} + 4X(5G_{4,XX} \\ & + G_{4,XXX} X))) + 6G_3(-X(G_4 + 2G_{4,X} X)(2(G_{3,XX} - 2G_{4,\phi XX}) X(G_4 + 2G_{4,X} X) \\ & + G_{3,X}(5G_4 - 2X(G_{4,X} + 4G_{4,XX} X)) + 2G_{4,\phi X}(-5G_4 + 2X(G_{4,X} + 4G_{4,XX} X))) \\ & + 2G_{4,\phi}(G_4^2 + 2X^2(5G_{4,X}^2 + 8G_{4,XX}^2 X^2 + 4G_{4,X} X(G_{4,XX} - G_{4,XXX} X)) \\ & - G_4 X(11G_{4,X} + 4X(5G_{4,XX} + G_{4,XXX} X)))) + 4((G_4 + 2G_{4,X} X)^2(G_{2,X}(G_4 + 2G_{4,X} X) \\ & - 2G_{3,\phi}(G_4 + 2G_{4,X} X) + X(-3(G_{3,X} - 2G_{4,\phi X})^2 X + 2(G_{2,XX} - G_{3,\phi X})(G_4 + 2G_{4,X} X))) \\ & + 3G_{4,\phi} X(G_4 + 2G_{4,X} X)(2(G_{3,XX} - 2G_{4,\phi XX}) X(G_4 + 2G_{4,X} X) + G_{3,X}(5G_4 - 2X(G_{4,X} + 4G_{4,XX} X)) \\ & + 2G_{4,\phi X}(-5G_4 + 2X(G_{4,X} + 4G_{4,XX} X))) - 3G_{4,\phi}^2(G_4^2 + 2X^2(5G_{4,X}^2 + 8G_{4,XX}^2 X^2 \\ & + 4G_{4,X} X(G_{4,XX} - G_{4,XXX} X)) - G_4 X(11G_{4,X} + 4X(5G_{4,XX} + G_{4,XXX} X)))))) \\ & + 24H(X((G_4 + 2G_{4,X} X)((2G_{3,X} - 5G_{4,\phi X}) G_4^2 + G_4(-(G_{3,X} + 2G_{4,\phi X}) G_{4,X} + (G_{3,XX} - 2G_{4,\phi XX}) G_4) X \\ & + 2((G_{3,X} - 4G_{4,\phi X}) G_{4,X}^2 + ((G_{3,XX} - 2G_{4,\phi XX}) G_{4,X} - 2(G_{3,X} - 2G_{4,\phi X}) G_{4,XX}) G_4) X^2) \\ & + G_3(-6G_{4,X} G_4^2 + 3G_4(G_{4,X}^2 - 3G_{4,XX} G_4) X - 2(3G_{4,X}^3 - 2G_{4,X} G_{4,XX} G_4 + G_{4,XXX} G_4^2) X^2 \\ & - 4(G_{4,X}^2 G_{4,XX} - 2G_{4,XX}^2 G_4 + G_{4,X} G_{4,XXX} G_4) X^3)) \\ & + G_{4,\phi}(-G_4^3 + 4G_{4,X}^2 X^3(G_{4,X} + 2G_{4,XX} X) - 2G_4 X^2(9G_{4,X}^2 + 8G_{4,XX}^2 X^2 + 4G_{4,X} X(G_{4,XX} - G_{4,XXX} X)) \\ & + 2G_4^2 X(3G_{4,X} + X(9G_{4,XX} + 2G_{4,XXX} X)))) \dot{\phi}. \end{aligned}$$